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GENERALIZED SASAKIAN-SPACE-FORMS

BY

PABLO ALEGRE

Department of Geometry and Topology, Faculty of Mathematics
University of Sevilla, Apdo. Correos 1160, 41080 - Sevilla, Spain

AND

DAVID E. BLAIR

Department of Mathematics, Michigan State University
East Lansing, MI 48824-1027, USA
e-mail: blair@math.msu.edu

AND

ALFONSO CARRIAZO*

Department of Geometry and Topology, Faculty of Mathematics
University of Sevilla, Apdo. Correos 1160, 41080 - Sevilla, Spain
e-mail: carriazo@us.es

ABSTRACT

Generalized Sasakian-space-forms are introduced and studied. Many examples of these manifolds are presented, by using some different geometric techniques such as Riemannian submersions, warped products or conformal and related transformations. New results on generalized complex-space-forms are also obtained.

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1. Introduction

In differential geometry, the curvature of a Riemannian manifold (M,g) plays a fundamental role, and, as is well known, the sectional curvatures of a manifold determine the curvature tensor R completely. For any point $p \in M$ and any plane section $\pi \subseteq T_pM$, the sectional curvature $K(\pi)$ is defined by $K(\pi) = g(R(X,Y)Y,X)$, where X, Y are orthonormal vector fields in π. In such a case, we also denote $K(\pi)$ by $K(X \wedge Y)$. A Riemannian manifold with constant sectional curvature c is called a **real-space-form**, and its curvature tensor satisfies the equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

Models for these spaces are the Euclidean spaces ($c = 0$), the spheres ($c > 0$) and the hyperbolic spaces ($c < 0$).

A similar situation can be found in the study of complex manifolds from a Riemannian point of view. If (M,J,g) is a Kaehlerian manifold with constant holomorphic sectional curvatures $K(X \wedge JX) = c$, then it is said to be a **complex-space-form** and it is well-known that its curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}. \end{aligned}$$

The models now are $\mathbf{C}^n$, $\mathbf{CP}^n$ and $\mathbf{CH}^n$, depending on $c = 0$, $c > 0$ or $c < 0$.

More generally, if the curvature tensor of an almost Hermitian manifold M satisfies

$$\begin{aligned} R(X,Y)Z =& F_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ F_2\{g(X,JZ)JY - g(Y,JZ)JX + 2g(X,JY)JZ\}, \end{aligned}$$

F_1, F_2 being differentiable functions on M, then M is said to be a **generalized complex-space-form** (see [16, 18]). In [16], an important obstruction for such a space was presented by F. Tricerri and L. Vanhecke: If M is connected, $\dim(M) \geq 6$, and F_2 is not identically zero, then M is a complex-space-form (in particular, F_1 and F_2 must be constant). Nevertheless, there are examples of 4-dimensional generalized complex-space-forms with non-constant functions, such as those given by Z. Olszak in [10]. Many other authors have studied these manifolds and their submanifolds.

On the other hand, **Sasakian-space-forms** play a similar role in contact metric geometry to that of complex-space-forms (see the preliminaries section for more details). For such a manifold, the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z =& \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &+ \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\qquad + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.1}$$

These spaces can also be modeled, depending on $c > -3$, $c = -3$ or $c < -3$. In this paper, we will study almost contact metric manifolds satisfying a similar equation, in which the constant quantities $(c+3)/4$ and $(c-1)/4$ are replaced by differentiable functions. We call such a space a **generalized Sasakian-space-form**.

After a section containing some background on almost contact metric geometry, we introduce generalized Sasakian-space-forms, give some examples and we prove some basic properties. For instance, we prove that every generalized Sasakian-space-form with a K-contact structure is a Sasakian manifold, and, if the dimension is ≥ 5, a Sasakian-space-form. We also study the possibility of a generalized Sasakian-space-form being a contact metric manifold. Next, we present two sections mainly devoted to giving more interesting examples of generalized Sasakian-space-forms, with non-constant functions. To do so, we use a wide variety of geometric constructions, such as Riemannian submersions, product manifolds, warped products, conformal transformations, D-homothetic deformations and D-conformal deformations. Moreover, we also obtain some further results on generalized complex-space-forms.

2. Preliminaries

In this section, we recall some definitions and basic formulas which we will use later. For more background on almost contact metric manifolds, we recommend the reference [2].

An odd-dimensional Riemannian manifold (M,g) is said to be an **almost contact metric manifold** if there exist on M a $(1,1)$ tensor field ϕ, a vector field ξ (called the **structure vector field**) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$.

Such a manifold is said to be a **contact metric** manifold if $d\eta = \Phi$, where $\Phi(X,Y) = g(X,\phi Y)$ is called the **fundamental 2-form** of M. If, in addition,

ξ is a Killing vector field, then M is said to be a K-**contact manifold**. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X\xi = -\phi X$, for any vector field X on M. In a K-contact manifold, we have

$$(\nabla_X\phi)Y = R(\xi, X)Y, \tag{2.1}$$

for any X, Y (cf. [2], pp. 92–94).

On the other hand, the almost contact metric structure of M is said to be **normal** if $[\phi,\phi](X,Y) = -2d\eta(X,Y)\xi$, for any X, Y, where $[\phi,\phi]$ denotes the Nijenhuis torsion of ϕ, given by $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X, Y]$ $-\phi[X,\phi Y]$.

A normal contact metric manifold is called a **Sasakian manifold**. It can be proved that a Sasakian manifold is K-contact, and that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.2}$$

for any X, Y. Moreover, for a Sasakian manifold the following equation holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.3}$$

In [12], J. A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold M is a **trans-Sasakian manifold** if there exist two functions α and β on M such that

$$(\nabla_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X), \tag{2.4}$$

for any X, Y on M. In particular, from (2.4) it is easy to see that the following equations hold for a trans-Sasakian manifold:

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$

$$\mathrm{d}\eta = \alpha\Phi. \tag{2.6}$$

In particular, if $\beta = 0$, M is said to be an α-**Sasakian manifold**. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha = 1$.

Another important kind of trans-Sasakian manifolds is that of **cosymplectic manifolds**, obtained for $\alpha = \beta = 0$. In fact, it can be proved that this definition is equivalent to M being normal with η and Φ closed forms; cosymplectic manifolds were defined this way in [1]. From (2.5) we have

$$\nabla_X\xi = 0, \tag{2.7}$$

which, in particular, implies that ξ is a Killing vector field for a cosymplectic manifold. Therefore, it is affine and hence

$$R(X,\xi)Y = \nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi = 0. \tag{2.8}$$

On the other hand, if $\alpha = 0$, M is said to be a β-**Kenmotsu manifold**. Kenmotsu manifolds, defined in [7], are particular examples with $\beta = 1$.

Actually, in [9], Marrero showed that a trans-Sasakian manifold of dimension greater than or equal to 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.

Given an almost contact metric manifold (M,ϕ,ξ,η,g), a ϕ-**section** of M at $p \in M$ is a section $\pi \subseteq T_pM$ spanned by a unit vector X_p orthogonal to ξ_p, and ϕX_p. The ϕ-**sectional curvature** of π is defined by $K(X \wedge \phi X) = R(X,\phi X;\phi X, X)$. A Sasakian manifold with constant ϕ-sectional curvature c is called a **Sasakian-space-form**. In such a case, its Riemann curvature tensor is given by equation (1.1).

Finally, let us point out that all the functions we will refer to during this paper will be differentiable functions on the corresponding manifolds.

3. Definition and basic properties

Given an almost contact metric manifold (M,ϕ,ξ,η,g), we say that M is a **generalized Sasakian-space-form** if there exist three functions f_1, f_2 and f_3 on M such that

$$\begin{aligned} R(X,Y)Z =& f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &\quad + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{3.1}$$

for any vector fields X,Y,Z on M, where R denotes the curvature tensor of M. In such a case, we will write $M(f_1,f_2,f_3)$.

This kind of manifold appears as a natural generalization of the well-known Sasakian-space-forms $M(c)$, which can be obtained as particular cases of generalized Sasakian-space-forms, by taking $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$. Moreover, we can also find some other trivial examples:

Example 3.1: A cosymplectic-space-form, i.e., a cosymplectic manifold with constant ϕ-sectional curvature c, is a generalized Sasakian-space-form with $f_1 = f_2 = f_3 = c/4$ (see, e.g., [8]).

Example 3.2: A Kenmotsu-space-form, i.e., a Kenmotsu manifold with constant ϕ-sectional curvature c, is a generalized Sasakian-space-form with $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$ (see [7]).

Example 3.3: An almost contact metric manifold is said to be an **almost** $C(\alpha)$-**manifold** [6] if its Riemann curvature tensor satisfies

$$\begin{aligned} R(X,Y,Z,W) =& R(X,Y,\phi Z,\phi W) \\ &+ \alpha\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &+ g(X,\phi Z)g(Y,\phi W) - g(X,\phi W)g(Y,\phi Z)\}, \end{aligned}$$

for any vector fields X,Y,Z,W on M, where α is a real number. Moreover, if such a manifold has constant ϕ-sectional curvature equal to c, then its curvature tensor is given by

$$\begin{aligned} R(X,Y)Z =& \frac{c+3\alpha^2}{4}\{g(Y,Z)X - g(X,Z)Y)\} \\ &+ \frac{c-\alpha^2}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ \frac{c-\alpha^2}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{3.2}$$

and so it is a generalized Sasakian-space-form with $f_1 = (c+3\alpha^2)/4$ and $f_2 = f_3 = (c-\alpha^2)/4$.

As we can see from the previous examples, we find generalized Sasakian-space-forms with very different structures. The following results give us more information about the relationship between the structure of such a manifold and the functions f_1, f_2, f_3. In this sense, we have the following theorem from [3], which we adapt to our notation:

THEOREM 3.4: *Let (M,ϕ,ξ,η,g) be a connected generalized Sasakian-space-form with $f_2 = f_3$ not identically zero. If $\dim(M) \geq 5$ and $g(X,\nabla_X\xi) = 0$ for any vector field X orthogonal to ξ, then f_1 and f_2 are constant functions and $f_1 - f_2 \geq 0$. Moreover, if $f_1 - f_2 = 0$, then (M,ϕ,ξ,η,g) is a cosymplectic-space-form and if $f_1 - f_2 = \alpha^2 > 0$ then (M,ϕ,ξ,η,g) or $(M,-\phi,\xi,\eta,g)$ is an α-Sasakian manifold with constant ϕ-sectional curvature c and curvature tensor satisfying* (3.2).

Now, we are interested in the study of the structure of generalized Sasakian-space-forms with $f_2 \neq f_3$, in general. We first recall a well-known fact (see, e.g., [2], p. 92).

LEMMA 3.5: *In a K-contact manifold, the sectional curvature of a plane section containing* ξ *is equal to* 1.

PROPOSITION 3.6: *Let* $M(f_1, f_2, f_3)$ *be a generalized Sasakian-space-form. If* M *is a K-contact manifold, then* $f_3 = f_1 - 1$.

Proof: It is just necessary to take into account that, from (3.1), $R(X, \xi, \xi, X) = f_1 - f_3$, for any unit vector field X orthogonal to ξ, and to apply Lemma 3.5. ∎

In particular, as every Sasakian manifold is a K-contact manifold, we obtain from the above proposition that if $M(f_1, f_2, f_3)$ is a Sasakian manifold, then $f_3 = f_1 - 1$. Moreover, we have:

THEOREM 3.7: *Every generalized Sasakian-space-form with a K-contact structure is a Sasakian manifold.*

Proof: Given a K-contact generalized Sasakian-space-form $M(f_1, f_2, f_3)$, equation (2.1) is satisfied. Hence a direct computation using (3.1) gives $(\nabla_X \phi)Y = (f_1 - f_3)(g(X,Y)\xi - \eta(Y)X)$, for any vector fields X, Y on M. But, by virtue of Proposition 3.6, $f_1 - f_3 = 1$, and so the above equation means that M is a Sasakian manifold. ∎

But, what is the situation if M is a contact metric manifold? To give an answer to this question, we need another result from [2], p. 92:

LEMMA 3.8: *A contact metric manifold* M^{2n+1} *is a* K*-contact manifold if and only if* $S(\xi, \xi) = 2n$*, where* S *denotes the Ricci curvature tensor.*

THEOREM 3.9: *Let* $M(f_1, f_2, f_3)$ *be a generalized Sasakian-space-form. If* M *is a contact metric manifold with* $f_3 = f_1 - 1$*, then it is a Sasakian manifold.*

Proof: From (3.1), it can be checked that $S(\xi, \xi) = 2n(f_1 - f_3) = 2n$, since $f_3 = f_1 - 1$. Therefore, by virtue of Lemma 3.8, we have that M is a K-contact manifold, and so it is a Sasakian manifold, by Theorem 3.7. ∎

The condition in the above theorem of M being a contact metric manifold is necessary. For instance, if $N(c)$ is a complex-space-form, and we consider the warped product $M = (-\pi/2, \pi/2) \times_f N$, with $f(t) = \cos t$, we will prove in Theorem 4.8 that M is a generalized Sasakian-space-form with functions

$$f_1 = \frac{c - 4\sin^2 t}{4\cos^2 t}, \quad f_2 = \frac{c}{4\cos^2 t}, \quad f_3 = \frac{c - 4\sin^2 t}{4\cos^2 t} - 1.$$

Therefore, in this case $f_3 = f_1 - 1$ but, as we will show in Proposition 4.9, M is a $(0, -\tan t)$ trans-Sasakian manifold.

Moreover, we can obtain more information about $f_1 - f_3$ for a contact metric manifold:

THEOREM 3.10: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. If M is a contact metric manifold, then $f_1 - f_3$ is constant on M.*

Proof: From (3.1), it is easy to see that $R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}$, for any vector fields X, Y on M. Therefore, we just conclude by using Theorem 10 of [13], which implies that $f_1 - f_3$ must be constant on the manifold. ∎

We now give some results concerning some identities satisfied by the curvature tensor of a generalized Sasakian-space-form. The first two propositions can be obtained directly from (3.1):

PROPOSITION 3.11: *The ϕ-sectional curvature of a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$.*

PROPOSITION 3.12: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. Then,*

$$\begin{aligned} R(X,Y,Z,W) - R(X,Y,\phi Z,\phi W) = \\ (f_1 - f_2)\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ + g(X,\phi Z)g(Y,\phi W) - g(X,\phi W)g(Y,\phi Z)\} \\ + (f_3 - f_2)\{\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\ + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)\}, \end{aligned}$$

for any vector fields X, Y, Z, W on M.

In particular, if $f_2 = f_3$ and $f_1 - f_2$ is equal to a constant α, then M is a $C(\alpha)$-manifold.

The following result is proved in [2], pp. 94–95:

LEMMA 3.13: *Let M be a Sasakian manifold. If we put*

$$\begin{aligned} \widetilde{P}(X,Y,Z,W) = & d\eta(X,Z)g(Y,W) - d\eta(X,W)g(Y,Z) \\ & - d\eta(Y,Z)g(X,W) + d\eta(Y,W)g(X,Z), \end{aligned}$$

then we have

$$R(X,Y,Z,\phi W) + R(X,Y,\phi Z,W) = -\widetilde{P}(X,Y,Z,W), \tag{3.3}$$

for any vector fields X, Y, Z, W *on* M, *and*

$$R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W), \tag{3.4}$$

$$R(X, \phi X, Y, \phi Y) = R(X, Y, X, Y) + R(X, \phi Y, X, \phi Y) - 2\widetilde{P}(X, Y, X, \phi Y), \tag{3.5}$$

for any X, Y, Z, W *orthogonal to* ξ.

Let us now denote

$$\begin{aligned} P(X, Y, Z, W) =& g(X, \phi Z)g(Y, W) - g(X, \phi W)g(Y, Z) \\ & - g(Y, \phi Z)g(X, W) + g(Y, \phi W)g(X, Z), \end{aligned}$$

for any vector fields X, Y, Z, W on M. In particular, if M is a contact metric manifold, $P = \widetilde{P}$. From (3.1) we get an equation similar to (3.3):

PROPOSITION 3.14: *Let* $M(f_1, f_2, f_3)$ *be a generalized Sasakian-space-form. Then,*

$$R(X, Y, Z, \phi W) + R(X, Y, \phi Z, W) = -(f_1 - f_2)P(X, Y, Z, W), \tag{3.6}$$

for any X, Y, Z, W *orthogonal to* ξ.

Hence, we can state the following result:

THEOREM 3.15: *Let* $M(f_1, f_2, f_3)$ *be a generalized Sasakian-space-form. If* M *is a Sasakian manifold, then* $f_2 = f_3 = f_1 - 1$.

Proof: The equality $f_2 = f_1 - 1$ is directly obtained from (3.3) and (3.6). On the other hand, $f_3 = f_1 - 1$ comes from Proposition 3.6. ∎

By using the above theorem, we can obtain an important consequence:

COROLLARY 3.16: *Let* $M(f_1, f_2, f_3)$ *be a connected generalized Sasakian-space-form. If* $\dim(M) \geq 5$ *and* M *is* K*-contact, then* M *is a Sasakian-space-form.*

Proof: From Theorem 3.7, we know that M is a Sasakian manifold, and so, by virtue of Theorem 3.15, $f_2 = f_3$. If f_2 is not identically zero, then by applying Theorem 3.4, we have that f_1 and f_2 (and so f_3) must be constant functions on M. Now, from Proposition 3.11 we deduce that the ϕ-sectional curvatures of M are constant, and therefore, M is a Sasakian-space-form.

On the other hand, if $f_2 = f_3 \equiv 0$, then from Theorem 3.15 we know that $f_1 = 1$ and so M is a real-space-form with constant sectional curvature 1. In particular, M is a Sasakian-space-form with constant ϕ-sectional curvature 1. ∎

Furthermore, the above result is also true if M is a contact metric manifold with $f_3 = f_1 - 1$, by virtue of Theorem 3.9. Therefore, to find connected generalized Sasakian-space-forms with non-constant functions, we must work with other classes of almost contact metric manifolds. We will see an ample number of examples in the following sections.

On the other hand, we can check that (3.4) holds for any generalized Sasakian-space-form, by making the corresponding calculations from (3.1):

PROPOSITION 3.17: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. Then, we have*

$$R(\phi X, \phi Y, \phi Z, \phi W) = R(X, Y, Z, W),$$

for any vector fields X, Y, Z, W orthogonal to ξ.

In a similar way, with respect to (3.5) we can prove:

PROPOSITION 3.18: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. Then, we have*

$$R(X, \phi X, Y, \phi Y) = R(X, Y, X, Y) + R(X, \phi Y, X, \phi Y) - 2(f_1 - f_2)P(X, Y, X, \phi Y),$$

for any vector fields X, Y orthogonal to ξ.

Also from (3.1) we can obtain a further result concerning curvature identities.

PROPOSITION 3.19: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. Then, the equation*

$$R(X, Y, Z, W) = R(\phi X, \phi Y, Z, W) + R(\phi X, Y, \phi Z, W) + R(\phi X, Y, Z, \phi W)$$

holds for any X, Y, Z, W if and only if $f_1 = f_3$.

In particular, if M is Sasakian, the above equation does not hold.

4. Examples and main results

In this section, we will show some different non-trivial examples of generalized Sasakian-space-forms.

Let (M, ϕ, ξ, η, g) be an almost contact metric manifold and (N, J, G) a Kaehlerian manifold, with dimensions $2n + 1$ and $2n$ respectively, and suppose there exists a Riemannian submersion

$$\pi\colon (M, \phi, \xi, \eta, g) \longrightarrow (N, J, G)$$

satisfying

$$\text{i) } \mathcal{V}_x = \operatorname{Span}\{\xi_x\}, \quad \text{ii) } (JX)^* = \phi X^*,$$

for any point $x \in M$ and any vector fields X, Y on N, where $\mathcal{V}_x$ is the vertical subspace at x and we denote by * the horizontal lift with respect to the connection η. Under these conditions we also have $g(X^*, Y^*) = G(X, Y)$. Let us denote by ∇ and ∇' the Riemannian connections associated with g and G, and by R and R' the corresponding Riemann curvature tensors.

PROPOSITION 4.1: *Under the above conditions, if M is an (α, β) trans-Sasakian manifold and $N(F_1, F_2)$ is a generalized complex-space-form, then we have*

(4.1)
$$\begin{aligned}
R(X^*,Y^*)Z^* &= (F_1 \circ \pi)\{g(Y^*,Z^*)X^* - g(X^*,Z^*)Y^*\} \\
&+ ((F_2 \circ \pi) - \alpha^2)\{g(X^*,\phi Z^*)\phi Y^* - g(Y^*,\phi Z^*)\phi X^* + 2g(X^*,\phi Y^*)\phi Z^*\} \\
&+ \alpha\beta\{g(X^*,\phi Z^*)Y^* - g(Y^*,\phi Z^*)X^* + 2g(X^*,\phi Y^*)Z^* \\
&\quad - g(X^*,Z^*)\phi Y^* + g(Y^*,Z^*)\phi X^*\} + \beta^2\{g(X^*,Z^*)Y^* - g(Y^*,Z^*)X^*\} \\
&- X^*(\alpha)g(Y^*,\phi Z^*)\xi - X^*(\beta)g(Y^*,Z^*)\xi \\
&+ Y^*(\alpha)g(X^*,\phi Z^*)\xi + Y^*(\beta)g(X^*,Z^*)\xi,
\end{aligned}$$

for any vector fields X, Y, Z on N.

Proof: By virtue of O'Neill's equations [11] and (2.5), we have

$$\nabla_{X^*}Y^* = (\nabla'_X Y)^* + \alpha g(\phi X^*, Y^*)\xi - \beta g(X^*, Y^*), \tag{4.2}$$

$$[X^*, Y^*] = [X, Y]^* + 2\alpha g(\phi X^*, Y^*)\xi, \tag{4.3}$$

for any X, Y vector fields on N. Then, (4.1) can be obtained with a long straightforward computation from (4.2)–(4.3) and the formula of the curvature tensor of a generalized complex-space-form, by taking also into account that $[Z^*, \xi] = 0$ and so $\nabla_\xi Z^* = \nabla_{Z^*}\xi$, for any vector field Z on N. ∎

In particular, M is a Sasakian manifold if and only if $\alpha = 1$ and $\beta = 0$. In such a case, (4.1) reduces to

$$\begin{aligned}
R(X^*,Y^*)Z^* =&(F_1 \circ \pi)\{g(Y^*,Z^*)X^* - g(X^*,Z^*)Y^*\} \\
&+ ((F_2 \circ \pi) - 1)\{g(X^*,\phi Z^*)\phi Y^* - g(Y^*,\phi Z^*)\phi X^* \\
&+ 2g(X^*,\phi Y^*)\phi Z^*\},
\end{aligned} \tag{4.4}$$

and we have:

THEOREM 4.2: *Under the above conditions, if M is a Sasakian manifold, then it is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with functions*

$$f_1 = F_1 \circ \pi, \quad f_2 = (F_2 \circ \pi) - 1, \quad f_3 = (F_1 \circ \pi) - 1.$$

Proof: Given X, Y, Z vector fields on M, we can write them as $X = \widetilde{X} + \eta(X)\xi$, $Y = \widetilde{Y} + \eta(Y)\xi$ and $Z = \widetilde{Z} + \eta(Z)\xi$, where $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ are horizontal vector fields on M. By choosing a local orthonormal frame of basic vector fields on M and by virtue of the linearity of both sides of equation (4.4), it can be proved that it is also satisfied by general horizontal vector fields, and so by $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$.

On the other hand, from (2.1), (2.2) and (2.3), and by using the linearity of the curvature tensor, we have

$$\begin{aligned} R(X,Y)Z =& R(\widetilde{X},\widetilde{Y})\widetilde{Z} - \eta(X)\eta(Z)\widetilde{Y} \\ & + \eta(Y)\eta(Z)\widetilde{X} - g(\widetilde{X},\widetilde{Z})\eta(Y)\xi + g(\widetilde{Y},\widetilde{Z})\eta(X)\xi. \end{aligned}$$

Hence, the above remark and a direct calculation give

$$\begin{aligned} R(X,Y)Z =& (F_1 \circ \pi)\{g(Y,Z)X - g(X,Z)Y\} \\ & + ((F_2 \circ \pi) - 1)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + ((F_1 \circ \pi) - 1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & \qquad + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

concluding the proof. ∎

COROLLARY 4.3: *Let $N(F_1, F_2)$ be a Kaehlerian generalized complex-space-form. If there exist a Sasakian manifold M and a Riemannian submersion $\pi: M \to N$ in the above conditions, then $F_1 = F_2$.*

Proof: By combining Theorem 3.15 and Theorem 4.2, we see that $F_1 \circ \pi = F_2 \circ \pi$, which implies $F_1 = F_2$, since π is onto. ∎

We have a similar theorem for cosymplectic manifolds, i.e., trans-Sasakian manifolds with $\alpha = \beta = 0$.

THEOREM 4.4: *Under the above conditions, if M is a cosymplectic manifold, then it is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with functions*

$$f_1 = F_1 \circ \pi, \quad f_2 = F_2 \circ \pi, \quad f_3 = F_1 \circ \pi.$$

Proof: It can be done in a similar way to that of Theorem 4.2, by now taking into account (2.7) and (2.8). ∎

In particular, if N is a Kaehlerian manifold, it is well-known that $M = N \times \mathbf{R}$ with its usual product almost contact metric structure (see [2], p. 77) is a cosymplectic manifold. Moreover, if we denote by π the natural projection from M into N, $\pi\colon M \to N$ is a Riemannian submersion satisfying all the above conditions. Therefore, we have the following corollary:

Corollary 4.5: *Let $N(F_1, F_2)$ be a Kaehlerian generalized complex-space-form. Then $M = N \times \mathbf{R}$ is a generalized Sasakian-space-form with functions*

$$f_1 = F_1 \circ \pi, \quad f_2 = F_2 \circ \pi, \quad f_3 = F_1 \circ \pi.$$

Now, we will obtain more examples of generalized Sasakian-space-forms by using warped products (see [11]).

Given an almost Hermitian manifold (N, J, G), the warped product $M = \mathbf{R} \times_f N$, where $f > 0$ is a function on $\mathbf{R}$, can be endowed with an almost contact metric structure (ϕ, ξ, η, g_f). In fact,

$$g_f = \pi^*(g_{\mathbf{R}}) + (f \circ \pi)^2 \sigma^*(G) \tag{4.5}$$

is the warped product metric, where π and σ are the projections from $\mathbf{R} \times N$ on $\mathbf{R}$ and N, respectively; $\phi(X) = (J\sigma_* X)^*$, for any vector field X on M, and $\xi = \partial/\partial t$, where t denotes the coordinate of $\mathbf{R}$.

We need the following two lemmas from [11]:

Lemma 4.6: *Let us consider $M = B \times_f F$ and denote by ∇, ∇^B and ∇^F the Riemannian connections on M, B and F. If X, Y are vector fields on B and V, W are vector fields on F, then:*

(1) *$\nabla_X Y$ is the lift of $\nabla^B_X Y$.*
(2) *$\nabla_X V = \nabla_V X = (Xf/f)V$.*
(3) *The component of $\nabla_V W$ normal to the fibers is $-(g_f(V,W)/f)\mathrm{grad}\, f$.*
(4) *The component of $\nabla_V W$ tangent to the fibers is the lift of $\nabla^F_V W$.*

Lemma 4.7: *Let $M = B \times_f F$ be a warped product, with Riemann curvature tensor R. Given fields X, Y, Z on B and U, V, W on F, then:*

(1) *$R(X,Y)Z$ is the lift of $R^B(X,Y)Z$.*
(2) *$R(V,X)Y = -(H^f(X,Y)/f)V$, where H^f is the Hessian of f.*
(3) *$R(X,Y)V = R(V,W)X = 0$.*
(4) *$R(X,V)W = -(g_f(V,W)/f)\nabla_X(\mathrm{grad}\, f)$.*
(5) *$R(V,W)U = R^F(V,W)U$*
$$+(g_f(\mathrm{grad}\, f, \mathrm{grad}\, f)/f^2)\{g_f(V,U)W - g_f(W,U)V\}.$$

THEOREM 4.8: *Let $N(F_1, F_2)$ be a generalized complex-space-form. Then, the warped product $M = \mathbf{R} \times_f N$, endowed with the almost contact metric structure (ϕ, ξ, η, g_f), is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with functions*

$$f_1 = \frac{(F_1 \circ \pi) - f'^2}{f^2}, \quad f_2 = \frac{F_2 \circ \pi}{f^2}, \quad f_3 = \frac{(F_1 \circ \pi) - f'^2}{f^2} + \frac{f''}{f}.$$

Proof: For any vector fields X, Y, Z on M, we can write $X = \eta(X)\xi + U$, $Y = \eta(Y)\xi + V$, $Z = \eta(Z)\xi + W$, where U, V, W are vector fields on N. Then, by virtue of Lemma 4.7, and by taking into account that $\mathbf{R}$ is flat, we have

$$\begin{aligned} R(X,Y)Z =& \eta(X)\eta(Z)\frac{H^f(\xi,\xi)}{f}V - \eta(Y)\eta(Z)\frac{H^f(\xi,\xi)}{f}U \\ & - \frac{g_f(V,W)}{f}\eta(X)\nabla_\xi \operatorname{grad}(f) + \frac{g_f(U,W)}{f}\eta(Y)\nabla_\xi \operatorname{grad}(f) \\ & + R^N(U,V)W + \frac{g_f(\operatorname{grad}(f),\operatorname{grad}(f))}{f^2}\{g_f(U,W)V - g_f(V,W)U\}. \end{aligned} \tag{4.6}$$

Let us first notice that $\operatorname{grad}(f) = f'\xi$, since $f = f(t)$. Therefore,

$$\nabla_\xi \operatorname{grad}(f) = f''\xi, \tag{4.7}$$

since, from Lemma 4.7, we know that $\nabla_\xi \xi = 0$. Moreover,

$$H^f(\xi,\xi) = g(\nabla_\xi \operatorname{grad}(f), \xi) = f'', \quad g_f(\operatorname{grad}(f), \operatorname{grad}(f)) = f'^2. \tag{4.8}$$

Now, from (4.5)–(4.8), and by using that N is a generalized complex-space-form, we have

$$\begin{aligned} R(X,Y)Z =& \frac{f''}{f}\{\eta(X)\eta(Z)V - \eta(Y)\eta(Z)U \\ & + f^2 g(U,W)\eta(Y)\xi - f^2 g(V,W)\eta(X)\xi\} \\ & + (F_1 \circ \pi)\{g(V,W)U - g(U,W)V\} \\ & + (F_2 \circ \pi)\{g(U,JW)JV - g(V,JW)JU + 2g(U,JV)JW\} \\ & + \left(\frac{f'}{f}\right)^2 \{f^2 g(U,W)V - f^2 g(V,W)U\}. \end{aligned} \tag{4.9}$$

Then, the proof can be finished with a straightforward calculation from (4.9), by taking into account (4.5) and the relationship between X, Y, Z and U, V, W. ∎

In particular, if $N(a, b)$ is a generalized complex-space-form with constant functions, then we have a generalized Sasakian-space-form

$$M\left(\frac{a - f'^2}{f^2}, \frac{b}{f^2}, \frac{a - f'^2}{f^2} + \frac{f''}{f}\right),$$

with non-constant functions. Moreover, if $N(c)$ is a complex-space-form, we obtain

$$M\Big(\frac{c-4f'^2}{4f^2}, \frac{c}{4f^2}, \frac{c-4f'^2}{4f^2}+\frac{f''}{f}\Big).$$

Hence, for example, the warped products $\mathbf{R}\times_f \mathbf{C}^n$, $\mathbf{R}\times_f \mathbf{CP}^n(4)$ and $\mathbf{R}\times_f \mathbf{CH}^n(-4)$ are generalized Sasakian-space-forms with functions

$$\begin{array}{lll} f_1 = -\frac{f'^2}{f^2}, & f_2 = 0, & f_3 = -\frac{f'^2}{f^2}+\frac{f''}{f}, \\ f_1 = \frac{1-f'^2}{f^2}, & f_2 = \frac{1}{f^2}, & f_3 = \frac{1-f'^2}{f^2}+\frac{f''}{f}, \\ f_1 = \frac{-1-f'^2}{f^2}, & f_2 = \frac{-1}{f^2}, & f_3 = \frac{-1-f'^2}{f^2}+\frac{f''}{f}, \end{array}$$

respectively.

Therefore, this method provides us with a wide range of examples of generalized Sasakian-space-forms with arbitrary dimensions and non-constant functions. Let us observe that, in general, $f_2 \neq f_3$ in the above examples, and so Theorem 3.4 does not impose any restriction in this case.

If $f=1$, the warped product $\mathbf{R}\times_f N$ is isometric to the usual Riemannian product $N\times\mathbf{R}$, and so we can deduce from Theorem 4.8 that the condition of N being Kaehlerian in Corollary 4.5 is not necessary, and that result is true for any generalized complex-space-form.

On the other hand, the following proposition gives us some information about the structure of these warped products:

PROPOSITION 4.9: *Let N be an almost Hermitian manifold. Then, $\mathbf{R}\times_f N$ is a $(0,\beta)$ trans-Sasakian manifold, with $\beta = f'/f$, if and only if N is a Kaehlerian manifold.*

Proof: By virtue of Lemma 4.6, a direct calculation gives

$$(\nabla_X\phi)Y = \frac{f'}{f}\{g_f(Y,\phi X)\xi - \eta(Y)\phi X\} + (\nabla^N_U J)V,$$

where we are using the same notation as in the proof of Theorem 4.8. Then, the proof ends by comparing this equation with (2.4). ∎

In particular, if $f=1$, we obtain that $N\times\mathbf{R}$ is cosymplectic if and only if N is a Kaehlerian manifold [5].

We conclude this section with a study of Bianchi's identities for a generalized Sasakian-space-form.

First, we can see from (3.1) that we do not obtain any special conditions on the functions of a generalized Sasakian-space-form from the first Bianchi identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

We then consider Bianchi's second identity,

$$\mathfrak{S}_{W,X,Y}(\nabla_W R)(X,Y)Z = 0,$$

where $\mathfrak{S}$ represents the cyclic sum on W, X, Y. If we denote

$$\begin{aligned}
R_1(X,Y)Z &= g(Y,Z)X - g(X,Z)Y,\\
R_2(X,Y)Z &= g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,\\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,
\end{aligned}$$

then, the second Bianchi identity for a generalized Sasakian-space-form looks like

$$\begin{aligned}
&\mathfrak{S}_{W,X,Y}\{W(f_1)R_1(X,Y)Z + f_2(\nabla_W R_2)(X,Y)Z + W(f_2)R_2(X,Y)Z\\
&\qquad + f_3(\nabla_W R_3)(X,Y)Z + W(f_3)R_3(X,Y)Z\} = 0,
\end{aligned} \tag{4.10}$$

since $(\nabla_W R_1)(X,Y)Z = 0$.

If we first put in (4.10) W, X, Y, Z vector fields on M orthogonal to ξ, then

$$\begin{aligned}
\mathfrak{S}_{W,X,Y}\{&W(f_1)\{g(Y,Z)X - g(X,Z)Y\}\\
&+ W(f_2)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\
&+ f_2\{g(X,(\nabla_W\phi)Z)\phi Y + g(X,\phi Z)(\nabla_W\phi)Y - g(Y,(\nabla_W\phi)Z)\phi X\\
&- g(Y,\phi Z)(\nabla_W\phi)X + 2g(X,(\nabla_W\phi)Y)\phi Z + 2g(X,\phi Y)(\nabla_W\phi)Z\}\\
&+ f_3\{g(X,Z)g(Y,\nabla_W\xi)\xi - g(Y,Z)g(X,\nabla_W\xi)\xi\}\} = 0.
\end{aligned}$$

Now, we choose unit vector fields X, Y such that Y is orthogonal to $X, \phi X$, and we put $Z = X$ and $W = \phi Y$. Then, by taking the inner product by ϕY and by ϕX, after a long calculation we obtain

$$Y(f_1) - 3f_2 g(\phi Y,(\nabla_X\phi)X) = 0, \tag{4.11}$$

$$-2X(f_2) + 3f_2\{g(\phi Y,(\nabla_Y\phi)X)) + g(X,(\nabla_{\phi Y}\phi)Y)\} = 0. \tag{4.12}$$

On the other hand, if we put in (4.10) $W = \xi$ and X, Y, Z orthogonal to ξ, we get

$$\begin{aligned}
&\xi(f_1)\{g(Y,Z)X - g(X,Z)Y\}\\
&\quad + \xi(f_2)\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\}\\
&\quad + f_2\{g(X,(\nabla_\xi\phi)Z)\phi Y + g(X,\phi Z)(\nabla_\xi\phi)Y - g(Y,(\nabla_\xi\phi)Z)\phi X\\
&\qquad - g(Y,\phi Z)(\nabla_\xi\phi)X + 2g(X,(\nabla_\xi\phi)Y)\phi Z + 2g(X,\phi Y)(\nabla_\xi\phi)Z\}\\
&\quad + f_3\{g(X,Z)g(Y,\nabla_\xi\xi)\xi - g(Y,Z)g(X,\nabla_\xi\xi)\xi\}
\end{aligned}$$

$$\begin{aligned}&- X(f_1)g(Y,Z)\xi + X(f_3)g(Y,Z)\xi\\&+ f_2\{g(Y,\phi Z)(\nabla_X\phi)\xi - g(\xi,(\nabla_X\phi)Z)\phi Y + 2g(Y,(\nabla_X\phi)\xi)\phi Z\}\\&+ f_3\{-g(Z,\nabla_X\xi)Y + g(Y,Z)\nabla_X\xi\} + Y(f_1)g(X,Z)\xi - Y(f_3)g(X,Z)\xi\\&+ f_2\{g(\xi,(\nabla_Y\phi)Z)\phi X - g(X,\phi Z)(\nabla_Y\phi)\xi + 2g(\xi,(\nabla_Y\phi)X)\phi Z\}\\&+ f_3\{g(Z,\nabla_Y\xi)X - g(X,Z)\nabla_Y\xi\} = 0.\end{aligned} \tag{4.13}$$

If we choose X and Y unit vector fields such that Y is orthogonal to $X, \phi X$, $Z = X$, and we multiply by ξ, we obtain

$$f_3 g(Y,\nabla_\xi\xi) + Y(f_1) - Y(f_3) = 0. \tag{4.14}$$

We put again $Z = X$, and we multiply (4.13) first, by Y and second, by ϕY. Then, we have

$$\xi(f_1) + f_3\{g(X,\nabla_X\xi) + g(\nabla_Y\xi, Y)\} = 0, \tag{4.15}$$

$$f_2 g(\xi,(\nabla_X\phi)X) + f_3 g(\nabla_Y\xi,\phi Y) = 0. \tag{4.16}$$

Now, we can prove the following result:

THEOREM 4.10: *Let $M(f_1,f_2,f_3)$ be a generalized Sasakian-space-form. If M is a $(0,\beta)$ trans-Sasakian manifold with $\dim(M)\geq 5$, then $X(f_i)=0$ for any X orthogonal to ξ, $i=1,2,3$, and the following equations hold:*

$$\xi(f_1) + 2\beta f_3 = 0, \tag{4.17}$$

$$\xi(f_2) + 2\beta f_2 = 0. \tag{4.18}$$

Proof: By using (2.4), we see that

$$g(\phi Y,(\nabla_X\phi)X) = g(\phi Y,\beta(g(\phi X,X)\xi - \eta(X)\phi X)) = 0,$$

for any vector field Y orthogonal to X,ξ, and

$$\begin{gathered}g(\phi Y,(\nabla_Y\phi)X)) + g(X,(\nabla_{\phi Y}\phi)Y) =\\ g(\phi Y,\beta(g(\phi Y,X)\xi - \eta(X)\phi Y)) - g(X,\beta(g(\phi^2 Y,Y)\xi - \eta(Y)\phi^2 Y)) = 0,\end{gathered}$$

for any X, Y orthogonal to ξ. Therefore, from (4.11) and (4.12) we deduce that $X(f_1) = X(f_2) = 0$, for any vector field X orthogonal to ξ.

On the other hand, since (2.5) implies that $\nabla_\xi\xi = 0$, from (4.14) we know that $X(f_1) - X(f_3) = 0$, for any X orthogonal to ξ. But, as $X(f_1) = 0$, we have that $X(f_3) = 0$.

Finally, by a similar way, we can obtain equations (4.17) and (4.18) from (4.15) and (4.16). ∎

For example, it is easy to see that the functions appearing in Theorem 4.8 satisfy equations (4.17) and (4.18).

In particular, if we consider $\beta = 0$ in the above theorem, we can state the following result for cosymplectic generalized Sasakian-space-forms:

COROLLARY 4.11: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form. If M is a connected cosymplectic manifold with $\dim(M) \geq 5$, then the functions f_1, f_2 are constant and f_3 only depends on the direction of ξ.*

We then obtain an important result for complex-space-forms as a consequence of Corollary 4.11:

COROLLARY 4.12: *Let $N(F_1, F_2)$ be a connected Kaehlerian generalized complex-space-form with $\dim(N) \geq 4$. Then, N is a complex-space-form.*

Proof: By virtue of Corollary 4.5, we know that $M = N \times \mathbf{R}$ is a generalized Sasakian-space-form with $f_1 = F_1 \circ \pi$ and $f_2 = F_2 \circ \pi$, where π denotes the natural projection from M on N. However, under these conditions, M is a connected cosymplectic manifold with dimension greater than or equal to 5. Therefore, Corollary 4.11 implies that f_1 and f_2 must be constant, and so F_1 and F_2, since π is onto.

On the other hand, it can be proved that the holomorphic sectional curvature of a generalized complex-space-form is given by $F_1 + 3F_2$, which is now a constant. Then, N is a complex-space-form. ∎

In fact, we can find this result, proved for generalized complex-space-form with dimension greater than or equal to 6 and F_2 non-identically zero, as Theorem 12.7 of [16]. In our case, the manifold has to be Kaehlerian, but the result also holds for dimension 4, and F_2 could be zero (in such a case, $F_1 = F_2 = 0$; the model of such a complex-space-form is $\mathbf{C}^n$). Therefore, Corollary 4.12 gives some additional knowledge.

Finally, let us notice that, from equations (4.17) and (4.18), we can get some information about the functions f_i of a generalized Sasakian-space-form with a $(0, \beta)$ trans-Sasakian structure.

Actually, by integrating with respect to t in (4.17), we deduce that, locally,

$$f_1 = \widetilde{F}_1 - 2 \int \beta f_3 dt,$$

where $\widetilde{F}_1$ is a function such that $\partial\widetilde{F}_1/\partial t = 0$.

Similarly, from (4.18) we have that, locally,

$$f_2 = \widetilde{F}_2 e^{-2\int \beta dt},$$

with $\widetilde{F}_2$ such that $\partial\widetilde{F}_2/\partial t = 0$.

5. Generalized Sasakian-space-forms and conformal changes of metric

Let (M,ϕ,ξ,η,g) be an almost contact metric manifold. We now consider a conformal change of metric

$$g^* = \rho^2 g, \tag{5.1}$$

where ρ is a positive function on M. We can easily prove that, if we put

$$\phi^* = \phi, \quad \xi^* = \frac{1}{\rho}\xi, \quad \eta^* = \rho\eta, \tag{5.2}$$

then $(M,\phi^*,\xi^*,\eta^*,g^*)$ is also an almost contact metric manifold [17].

It is well-known (for instance, see [4]) that, if we denote by ∇^* the Riemannian connection associated with g^* and by R^* the curvature tensor of g^*, then we have

$$\begin{aligned} \nabla^*_X Y =& \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X,Y)U, \\ R^*(X,Y)Z =& R(X,Y)Z - t(Y,Z)X + t(X,Z)Y \\ &- g(Y,Z)TX + g(X,Z)TY, \end{aligned} \tag{5.3}$$

for vector fields X, Y, Z on M, where

$$\begin{aligned} U = \operatorname{grad}(\log(\rho)), \quad \omega = \mathrm{d}(\log(\rho)), \\ TX = \nabla_X U - \omega(X)U + \frac{1}{2}\omega(U)X, \end{aligned} \tag{5.4}$$

$$t(X,Y) = (\nabla_X\omega)Y - \omega(X)\omega(Y) + \frac{1}{2}\omega(U)g(X,Y). \tag{5.5}$$

Moreover, from (5.4) and (5.5) we get

$$\begin{aligned} -t(Y,Z)X + t(X,Z)Y - g(Y,Z)TX + g(X,Z)TY = \\ -\omega(U)\{g(Y,Z)X - g(X,Z)Y\} \\ -\{\omega(X)\omega(Z)Y - \omega(Y)\omega(Z)X + g(X,Z)\omega(Y)U - g(Y,Z)\omega(X)U\} \\ + g(\nabla_X U,Z)Y - g(\nabla_Y U,Z)X + g(X,Z)\nabla_Y U - g(Y,Z)\nabla_X U. \end{aligned} \tag{5.6}$$

Hence, if (M,ϕ,ξ,η,g) if a generalized Sasakian-space-form, with functions f_1, f_2, f_3, by using (3.1), (5.1), (5.2), (5.3) and (5.6), we can write

$$
\begin{aligned}
R^*(X,Y)Z = {} & \frac{f_1-\omega(U)}{\rho^2}\{g^*(Y,Z)X-g^*(X,Z)Y\} \\
& + \frac{f_2}{\rho^2}\{g^*(X,\phi^*Z)\phi^*Y-g^*(Y,\phi^*Z)\phi^*X+2g^*(X,\phi^*Y)\phi^*Z\} \\
& + \frac{f_3}{\rho^2}\{\eta^*(X)\eta^*(Z)Y-\eta^*(Y)\eta^*(Z)X \\
& \qquad + g^*(X,Z)\eta^*(Y)\xi^*-g^*(Y,Z)\eta^*(X)\xi^*\} \\
& - \{\omega(X)\omega(Z)Y-\omega(Y)\omega(Z)X+g(X,Z)\omega(Y)U-g(Y,Z)\omega(X)U\} \\
& + g(\nabla_X U,Z)Y-g(\nabla_Y U,Z)X+g(X,Z)\nabla_Y U-g(Y,Z)\nabla_X U,
\end{aligned}
\tag{5.7}
$$

for vector fields X,Y,Z.

Let us now suppose that there exists a function μ on M such that $U=\mu\xi$, which implies that $\omega=\mu\eta$ and $\omega(U)=\mu^2$. Then,

$$
\begin{aligned}
& \omega(X)\omega(Z)Y-\omega(Y)\omega(Z)X+g(X,Z)\omega(Y)U-g(Y,Z)\omega(X)U = \\
& \frac{\mu^2}{\rho^2}\{\eta^*(X)\eta^*(Z)Y-\eta^*(Y)\eta^*(Z)X+g^*(X,Z)\eta^*(Y)\xi^*-g^*(Y,Z)\eta^*(X)\xi^*\}.
\end{aligned}
\tag{5.8}
$$

On the other hand, $\nabla_X U=\mu\nabla_X\xi+X(\mu)\xi$, for any vector field X, and clearly, $g(\nabla_X\xi,\xi)=0$. Therefore, we will suppose in addition that the function μ is a constant and that there exists a function β on M such that $\nabla_X\xi=\beta(X-\eta(X)\xi)$, for any X. Hence, $\nabla_X U=\mu\beta(X-\eta(X)\xi)$ and

$$
\begin{aligned}
g(\nabla_X U,Z)Y-g(\nabla_Y U,Z)X+g(X,Z)\nabla_Y U-g(Y,Z)\nabla_X U = {} & \\
-\frac{2\mu\beta}{\rho^2}\{g^*(Y,Z)X-g^*(X,Z)Y\} & \\
-\frac{\mu\beta}{\rho^2}\{\eta^*(X)\eta^*(Z)Y-\eta^*(Y)\eta^*(Z)X & \\
+g^*(X,Z)\eta^*(Y)\xi^*-g^*(Y,Z)\eta^*(X)\xi^*\}. &
\end{aligned}
\tag{5.9}
$$

From (5.7), (5.8) and (5.9), we get

$$
\begin{aligned}
R^*(X,Y)Z = {} & \frac{f_1-\mu^2-2\mu\beta}{\rho^2}\{g^*(Y,Z)X-g^*(X,Z)Y\} \\
& +\frac{f_2}{\rho^2}\{g^*(X,\phi^*Z)\phi^*Y-g^*(Y,\phi^*Z)\phi^*X+2g^*(X,\phi^*Y)\phi^*Z\} \\
& +\frac{f_3-\mu^2-\mu\beta}{\rho^2}\{\eta^*(X)\eta^*(Z)Y-\eta^*(Y)\eta^*(Z)X \\
& \qquad +g^*(X,Z)\eta^*(Y)\xi^*-g^*(Y,Z)\eta^*(X)\xi^*\}.
\end{aligned}
$$

We have then proved the following result:

THEOREM 5.1: *Let $M(f_1, f_2, f_3)$ be a generalized Sasakian-space-form such that there exists a function β on M satisfying $\nabla_X\xi = \beta(X - \eta(X)\xi)$, for any vector field X. Let us consider a positive function ρ on M and the almost contact metric structure given by (5.2). Let $U = \mathrm{grad}(\log(\rho))$ and $\omega = \mathrm{d}(\log(\rho))$ be. If there exists a constant $k \neq 0$ such that $U = k\xi$, then $(M, \phi^*, \xi^*, \eta^*, g^*)$ is a generalized Sasakian-space-form $M(f_1^*, f_2^*, f_3^*)$ with functions*

$$f_1^* = \frac{f_1 - k^2 - 2k\beta}{\rho^2}, \quad f_2^* = \frac{f_2}{\rho^2}, \quad f_3^* = \frac{f_3 - k^2 - k\beta}{\rho^2}.$$

We now show that we can find almost contact metric manifolds satisfying all the above conditions.

Example 5.2: Let $N(c)$ be a complex-space-form and put $M = \mathbf{R} \times_f N$, with $f = f(t) > 0$. If we consider on the warped product M the almost contact metric structure described above, we already know that it is a generalized Sasakian-space-form $M(f_1, f_2, f_3)$ with functions

$$f_1 = \frac{c - 4f'^2}{4f^2}, \quad f_2 = \frac{c}{4f^2}, \quad f_3 = \frac{c - 4f'^2}{4f^2} + \frac{f''}{f}.$$

Moreover, from Proposition 4.9, we also know that it is a $(0, \beta)$ trans-Sasakian manifold with $\beta = f'/f$, and so

$$\nabla_X\xi = \frac{f'}{f}(X - \eta(X)\xi),$$

for any X.

On the other hand, if we put $\rho = \rho(t) > 0$, then

$$U = (\log(\rho))'\frac{\partial}{\partial t} = \frac{\rho'}{\rho}\xi,$$

and so there exists a constant k such that $U = k\xi$ if and only if $\rho(t) = Ke^{kt}$, where $K > 0$ is a constant. In such a case, by virtue of Theorem 5.1, $(M, \phi^*, \xi^*, \eta^*, g^*)$ is a generalized Sasakian-space-form with functions

$$f_1^* = \frac{c - 4(f' + kf)^2}{(2Kfe^{kt})^2}, \quad f_2^* = \frac{c}{(2Kfe^{kt})^2},$$
$$f_3^* = \frac{c - 4(f' + kf)^2 + 4f(kf' + f'')}{(2Kfe^{kt})^2}.$$

Actually, we can get some other examples of generalized Sasakian-space-forms obtained through conformal changes of metrics, without the assumption of μ

being a constant. In fact, if $N(c)$ is a complex-space-form, $M = \mathbf{R}\times_f N(c)$ and we choose $\rho=\rho(t)$, we have already pointed out that $U=(\rho'/\rho)\xi$, and so $\mu=\rho'/\rho$. Moreover, it can be proved that $X(\rho'/\rho)=\eta(X)(\log(\rho))''$, for any vector field X on M, and so

$$\nabla_X U=\frac{\rho' f'}{\rho f}(X-\eta(X)\xi)+\eta(X)(\log(\rho))''\xi. \tag{5.10}$$

Then, from (5.7), (5.8) and (5.10), we obtain

$$\begin{aligned} R^*(X,Y)Z &= \frac{1}{\rho^2}\Big(\frac{c}{4f^2}-\Big(\frac{f'}{f}+\frac{\rho'}{\rho}\Big)^2\Big)\{g^*(Y,Z)X-g^*(X,Z)Y\}\\ &+\frac{1}{\rho^2}\frac{c}{4f^2}\{g^*(X,\phi^*Z)\phi^*Y-g^*(Y,\phi^*Z)\phi^*X+2g^*(X,\phi^*Y)\phi^*Z\}\\ &+\frac{1}{\rho^2}\Big(\frac{c}{4f^2}-\Big(\frac{f'}{f}+\frac{\rho'}{\rho}\Big)^2+\frac{\rho' f'}{\rho f}+\frac{f''}{f}+(\log(\rho))''\Big)\\ &\cdot\{\eta^*(X)\eta^*(Z)Y-\eta^*(Y)\eta^*(Z)X+g^*(X,Z)\eta^*(Y)\xi^*-g^*(Y,Z)\eta^*(X)\xi^*\}, \end{aligned}$$

which implies the following theorem.

THEOREM 5.3: *Given a complex-space-form $N(c)$ and two positive functions $f=f(t)$, $\rho=\rho(t)$, the conformal change of metric with function ρ endows the warped product $M=\mathbf{R}\times_f N(c)$ with the structure of a generalized Sasakian-space-form $M(f_1^*,f_2^*,f_3^*)$ with functions*

$$f_1^*=\frac{1}{\rho^2}\Big(\frac{c}{4f^2}-\Big(\frac{f'}{f}+\frac{\rho'}{\rho}\Big)^2\Big),\quad f_2^*=\frac{1}{\rho^2}\frac{c}{4f^2},$$
$$f_3^*=\frac{1}{\rho^2}\Big(\frac{c}{4f^2}-\Big(\frac{f'}{f}+\frac{\rho'}{\rho}\Big)^2+\frac{\rho' f'}{\rho f}+\frac{f''}{f}+(\log(\rho))''\Big).$$

Moreover, there are other very useful metric transformations in contact Riemannian geometry, such as the D-homothetic deformations. Given an almost contact metric manifold (M,ϕ,ξ,η,g), such a deformation is defined by

$$\phi^*=\phi,\quad \xi^*=\frac{1}{a}\xi,\quad \eta^*=a\eta,\quad g^*=ag+a(a-1)\eta\otimes\eta, \tag{5.11}$$

where a is a positive constant (see [15]). It is clear that $(M,\phi^*,\xi^*,\eta^*,g^*)$ is also an almost contact metric manifold.

Let us suppose that (M,ϕ,ξ,η,g) is a $(0,\beta)$ trans-Sasakian manifold and put $a\neq 1$ to have a non-trivial D-homothetic deformation. Computing the Riemannian connection ∇^* of g^* and using equations (2.5)–(2.6), it can be proved that

$$\nabla^*_X Y=\nabla_X Y+\frac{a-1}{a}\beta g(\phi X,\phi Y)\xi. \tag{5.12}$$

Therefore, by using also (2.4) we can see that the corresponding curvature tensors satisfy the equation

$$\begin{aligned} R^*(X,Y)Z =& R(X,Y)Z \\ &+ \frac{a-1}{a}\{X(\beta)g(\phi Y,\phi Z)\xi - Y(\beta)g(\phi X,\phi Z)\xi\} \\ &+ \frac{a-1}{a}\beta^2\{g(\phi Y,\phi Z)X - g(\phi X,\phi Z)Y\}, \end{aligned} \tag{5.13}$$

for any vector fields X, Y, Z on M. Moreover, if M is a generalized Sasakian-space-form with functions f_1, f_2, f_3, from (3.1), (5.11), (5.13) and a direct computation, we get

$$\begin{aligned} R^*(X,Y)Z =& \Big(\frac{f_1}{a} + \frac{a-1}{a^2}\beta^2\Big)\{g^*(Y,Z)X - g^*(X,Z)Y\} \\ &+ \frac{f_2}{a}\{g^*(X,\phi^*Z)\phi^*Y - g^*(Y,\phi^*Z)\phi^*X + 2g^*(X,\phi^*Y)\phi^*Z\} \\ &+ \frac{f_3 + (f_1+\beta^2)(a-1)}{a^2}\{\eta^*(X)\eta^*(Z)Y - \eta^*(Y)\eta^*(Z)X\} \\ &+ \frac{f_3}{a}\{g^*(X,Z)\eta^*(Y)\xi^* - g^*(Y,Z)\eta^*(X)\xi^*\} \\ &+ \frac{a-1}{a^2}\{Y(\beta)\eta^*(X)\eta^*(Z)\xi^* - X(\beta)\eta^*(Y)\eta^*(Z)\xi^*\} \\ &- \frac{a-1}{a}\{Y(\beta)g^*(X,Z)\xi^* - X(\beta)g^*(Y,Z)\xi^*\}. \end{aligned} \tag{5.14}$$

Hence, if β is a constant, the last two lines in the above equation vanish, and so $(M, \phi^*, \xi^*, \eta^*, g^*)$ is a generalized Sasakian-space-form if and only if

$$\frac{f_3 + (f_1+\beta^2)(a-1)}{a^2} = \frac{f_3}{a},$$

which is equivalent to

$$f_1 - f_3 + \beta^2 = 0. \tag{5.15}$$

For example, let M be a warped product $\mathbf{R} \times_f N(c)$, $N(c)$ being a complex-space-form. In such a case, M is a $(0,\beta)$ trans-Sasakian manifold with $\beta = f'/f$, and so β equals a constant k if and only if $f(t) = Ke^{kt}$, $K > 0$, which implies that $f''/f = k^2$. But, from Theorem 4.8, we know that $f_3 = f_1 + f''/f$. Therefore, equation (5.15) is satisfied and, by virtue of (5.14), we can state the following theorem:

THEOREM 5.4: *Given a complex-space-form $N(c)$, a positive constant a and the function $f(t) = Ke^{kt}$, $k \in \mathbf{R}$, $K > 0$, the D-homothetic deformation with* *constant a endows the warped product $M = \mathbf{R} \times_f N(c)$ with the structure of a generalized Sasakian-space-form $M(f_1^*, f_2^*, f_3^*)$ with functions*

$$f_1^* = \frac{c}{4aK^2e^{2kt}} - \left(\frac{k}{a}\right)^2, \quad f_2^* = f_3^* = \frac{c}{4aK^2e^{2kt}}.$$

Notice that if $N(c)$ is a connected complex-space-form with $\dim(N) \geq 4$ and $c \neq 0$, then the almost contact metric manifold $(M, \phi^*, \xi^*, \eta^*, g^*)$ obtained above is a connected generalized Sasakian-space-form with $\dim(M) \geq 5$ and functions $f_2^* = f_3^*$ non-identically zero. Therefore, if

$$g^*(X, \nabla_X^* \xi^*) = 0, \tag{5.16}$$

for any vector field X orthogonal to ξ^*, then by virtue of Theorem 3.4, f_1^* and f_2^* should be constant functions. But, from (2.5), (5.11) and (5.12), a direct calculation shows that equation (5.16) can be written as $kg(X, X) = 0$, and so it holds if and only if $k = 0$. And, obviously, in such a case the above functions f_1^* and f_2^* are constant and then M is a cosymplectic-space-form.

On the other hand, if we consider now a warped product $M = \mathbf{R} \times_f N(c)$, with non-constant function $\beta = f'/f$, and we apply a D-homothetic deformation given by (5.11), by taking into account that $X(\beta) = 1/a\, \eta^*(X)\beta'$, equation (5.14) becomes

$$\begin{aligned} R^*(X,Y)Z &= \left(\frac{f_1}{a} + \frac{a-1}{a^2}\beta^2\right)\{g^*(Y,Z)X - g^*(X,Z)Y\} \\ &+ \frac{f_2}{a}\{g^*(X,\phi^*Z)\phi^*Y - g^*(Y,\phi^*Z)\phi^*X + 2g^*(X,\phi^*Y)\phi^*Z\} \\ &+ \frac{f_3 + (f_1+\beta^2)(a-1)}{a^2}\{\eta^*(X)\eta^*(Z)Y - \eta^*(Y)\eta^*(Z)X\} \\ &+ \frac{af_3 - (a-1)\beta'}{a^2}\{g^*(X,Z)\eta^*(Y)\xi^* - g^*(Y,Z)\eta^*(X)\xi^*\}. \end{aligned} \tag{5.17}$$

Therefore, $(M, \phi^*, \xi^*, \eta^*, g^*)$ is now a generalized Sasakian-space-form if and only if $f_3 + (f_1+\beta^2)(a-1) = af_3 - (a-1)\beta'$, which is equivalent to $f_3 - f_1 = \beta^2 + \beta'$. But, by virtue of Theorem 4.8, it is easy to see that this equation is always satisfied, and we have proved the following result:

THEOREM 5.5: *Given a complex-space-form $N(c)$, a positive constant a and a function $f = f(t) > 0$, the D-homothetic deformation with constant a endows the warped product $M = \mathbf{R} \times_f N(c)$ with the structure of a generalized Sasakian-space-form $M(f_1^*, f_2^*, f_3^*)$ with functions*

$$f_1^* = \frac{ac - 4f'^2}{4a^2f^2}, \quad f_2^* = \frac{c}{4af^2}, \quad f_3^* = \frac{ac - 4f'^2}{4a^2f^2} + \frac{f''}{a^2f}.$$

More generally, we can also consider a warped product $M = \mathbf{R} \times_f N(c)$, $N(c)$ being a complex-space-form, a function $\delta = \delta(t) > 0$ and the structure change given by

$$\phi^* = \phi, \quad \xi^* = \frac{1}{\delta^2}\xi, \quad \eta^* = \delta^2\eta, \quad g^* = \delta^2 g + \delta^2(\delta^2 - 1)\eta \otimes \eta. \tag{5.18}$$

Such a deformation, which clearly generalizes the D-homothetic ones, is a particular case of the so-called D-conformal deformations (see [14]).

In such a case, $U = \mathrm{grad}(\log(\delta)) = \delta'/\delta\ \xi$, $\omega = \mathrm{d}(\log(\delta)) = \delta'/\delta\ \eta$ and a direct computation gives

$$\nabla^*_X Y = \nabla_X Y + \Big(\beta\frac{\delta^2 - 1}{\delta^2} - \frac{\delta'}{\delta}\frac{1}{\delta^2}\Big) g(\phi X, \phi Y)\xi + \frac{\delta'}{\delta}(\eta(Y)X + \eta(X)Y), \tag{5.19}$$

for any vector fields X, Y, where ∇^* (resp. ∇) denotes the Riemannian connection associated with g^* (resp. g). Similarly to the previous cases studied in this section, by virtue of (2.4)–(2.6), (5.18), (5.19) and a straightforward calculation, we obtain

$$\begin{aligned}
R^*(X,Y)Z = {} & \frac{1}{\delta^2}\Big(f_1 - \frac{\delta'}{\delta}\beta + F\beta + F\frac{\delta'}{\delta}\Big)\{g^*(Y,Z)X - g^*(X,Z)Y\} \\
& + \frac{f_2}{\delta^2}\{g^*(X,\phi^*Z)\phi^*Y - g^*(Y,\phi^*Z)\phi^*X + 2g^*(X,\phi^*Y)\phi^*Z\} \\
& + \frac{1}{\delta^2}\Big(\frac{\delta^2-1}{\delta^2}f_1 + \frac{1}{\delta^2}f_3 - \frac{\delta'}{\delta}\beta + F\beta + F\frac{\delta'}{\delta} + \frac{1}{\delta^2}\Big(\Big(\frac{\delta'}{\delta}\Big)' - \Big(\frac{\delta'}{\delta}\Big)^2\Big)\Big) \\
& \cdot \{\eta^*(X)\eta^*(Z)Y - \eta^*(Y)\eta^*(Z)X\} \\
& + \frac{1}{\delta^2}(f_3 - F')\{g^*(X,Z)\eta^*(Y)\xi^* - g^*(Y,Z)\eta^*(X)\xi^*\},
\end{aligned}$$

where

$$\beta = \frac{f'}{f} \quad \text{and} \quad F = \beta\frac{\delta^2 - 1}{\delta^2} - \frac{\delta'}{\delta}\frac{1}{\delta^2}.$$

Therefore, $(M, \phi^*, \xi^*, \eta^*, g^*)$ is a generalized Sasakian-space-form if and only if

$$\frac{\delta^2-1}{\delta^2}f_1 + \frac{1}{\delta^2}f_3 - \frac{\delta'}{\delta}\beta + F\beta + F\frac{\delta'}{\delta} + \frac{1}{\delta^2}\Big(\Big(\frac{\delta'}{\delta}\Big)' - \Big(\frac{\delta'}{\delta}\Big)^2\Big) = f_3 - F',$$

which, by virtue of Theorem 4.8, is equivalent to

$$(\beta^2 + \beta')\frac{\delta^2 - 1}{\delta^2} = \frac{f''}{f}\frac{\delta^2 - 1}{\delta^2}.$$

But it is easy to see that the above equation is always satisfied, just by taking into account the definition of β. Hence, by using again Theorem 4.8, we have proved a new theorem, which generalizes Theorem 5.5:

THEOREM 5.6: *Given a complex-space-form $N(c)$ and two functions $f=f(t)>0$, $\delta=\delta(t)>0$, the D-conformal deformation with function δ endows the warped product $M=\mathbf{R}\times_f N(c)$ with the structure of a generalized Sasakian-space-form $M(f_1^*,f_2^*,f_3^*)$ with functions*

$$f_1^* = \frac{1}{\delta^2}\Big(\frac{c}{4f^2} - \frac{1}{\delta^2}\Big(\frac{f'}{f}+\frac{\delta'}{\delta}\Big)^2\Big), \quad f_2^* = \frac{c}{4\delta^2 f^2},$$

$$f_3^* = \frac{1}{\delta^2}\Big(\frac{c}{4f^2} - \frac{1}{\delta^2}\Big(\Big(\frac{f'}{f}+\frac{\delta'}{\delta}\Big)^2 - \Big(\frac{f''}{f}+\frac{\delta''}{\delta}\Big) + 2\Big(\frac{\delta'}{\delta}\Big)^2\Big)\Big).$$

Another useful change of metric for almost contact metric manifolds, similar to that of D-conformal deformations, is given by $g^* = \gamma^2 g + (1-\gamma^2)\eta\otimes\eta$, γ being a positive function on M. It is well-known that if (M,ϕ,ξ,η,g) is an almost contact metric manifold, then (M,ϕ,ξ,η,g^*) is also an almost contact metric manifold (for example, see [9]). Therefore, we could consider such a deformation acting on a warped product $M=\mathbf{R}\times_f N(c)$, with $\gamma=\gamma(t)$. But, in this situation, it is easy to see that M is just transformed into another warped product, with function γf, and so we would not obtain new significant examples of generalized Sasakian-space-forms in this way.

Finally, let us notice that if $N(c)$ is $\mathbf{C}^n$, $\mathbf{CP}^n$ or $\mathbf{CH}^n$, then we can obtain particular examples of generalized Sasakian-space-forms from Example 5.2 and Theorems 5.3, 5.4, 5.5 and 5.6, with the corresponding expressions for the curvature tensor of M, just by putting $c=0$, $c=4$ or $c=-4$ above, respectively.

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